

ON THE COMPUTATION OF RIGHT DERIVED FUNCTOR OF α -SECTION FUNCTOR

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Abstract

In this paper, we give several ways to compute generalized local cohomology modules, specially usual local cohomology modules, by using Gorenstein injective resolutions. Also, we find some bounds for vanishing of generalized local cohomology modules.

1. Introduction

The local cohomology theory of Grothendieck [10] has proved to be an important tool in algebraic geometry, commutative algebra, and algebraic topology.

Recently, some authors have studied the theory of local cohomology by using Gorenstein homological tool.

Enochs and Jenda in [7] defined Gorenstein injective modules and also they introduced Gorenstein flat modules in their joint work with

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Torrecillas [6]. Next, Sazeedeh [14]; Divaani-Aazar and Hajikarimi [5] studied the connection between local cohomology and Gorenstein injective modules.

The aim of this article is to give some ways for computing generalized local cohomology modules by using Gorenstein homological resolutions. More precisely, let R be a Noetherian ring and \mathfrak{a} be an ideal of R . Assume that $X \in D_{\mathfrak{I}}(R)$ and $Y \in D_{\mathfrak{I}}(R)$ are two nonhomologically trivial complexes. Assume that P is a projective resolution of X and I is a bounded to the left complex of Gorenstein injective R -modules such that $I \simeq Y$. Then, we prove that $R\Gamma_{\mathfrak{a}}(R\text{Hom}_R(X, Y)) \simeq \Gamma_{\mathfrak{a}}(\text{Hom}_R(P, I))$.

This result extends Sazeedeh's results [14, Section 3] to generalized local cohomology modules of complexes.

In the sequel, we give several ways for computing generalized local cohomology modules and we deduce some bounds for vanishing of generalized local cohomology modules.

2. Prerequisites

Throughout this paper, R is a commutative Noetherian ring with nonzero identity and \mathfrak{a} is an ideal of R .

2.1. Hyperhomology

Throughout, we will work within $D(R)$, the derived category of R -modules. The objects in $D(R)$ are complexes of R -modules and symbol \simeq denotes isomorphisms in this category. For a complex X , its supremum and infimum are defined, respectively, by $\text{sup}(X) := \sup\{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$ and $\text{inf}(X) := \inf\{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$, with the usual convention that $\text{sup}(\Phi) = -\infty$, $\text{inf}(\Phi) = \infty$.

Modules can be considered as complexes concentrated in degree zero. The full subcategory of complexes homologically bounded to the right (resp., left) is denoted by $D_{\downarrow}(R)$ (resp., $D_{\uparrow}(R)$). Also, the full subcategory of homologically bounded complexes will be denoted by $D_{\square}(R)$. We denote the full subcategory of homologically bounded (resp., bounded to the right) complexes with finitely generated homology modules by $D_{\square}^f(R)$ (resp., $D_{\downarrow}^f(R)$). For any complex X in $D_{\downarrow}(R)$ (resp., $D_{\uparrow}(R)$), there is a bounded to the right (resp., left) complex P (resp., I) of projective (resp., injective) R -modules, which is isomorphic to X in $D(R)$. A such complex P (resp., I) is called projective (resp., injective) resolution of X . Also, for any complex X in $D_{\downarrow}(R)$, there is a bounded to the right complex F of flat R -modules, which is isomorphic to X in $D(R)$. A such complex F is called a flat resolution of X .

2.2. Gorenstein dimension

An R -module N is said to be Gorenstein injective, if there exists an exact complex I of injective R -modules such that $N \cong \text{im}(I_1 \rightarrow I_0)$ and $\text{Hom}_R(E, I)$ is exact for all injective R -modules E .

Also, an R -module M is said to be Gorenstein flat, if there exists an exact sequence F of flat R -modules such that $M \cong \text{im}(F_0 \rightarrow F_{-1})$ and that $F \otimes_R E$ is exact for all injective R -modules E .

For a complex $X \in D_{\uparrow}(R)$, its Gorenstein injective dimension is defined as:

$$\text{Gid}_R X := \inf \{ \sup \{ l \in \mathbb{Z} \mid E_{-l} \neq 0 \} \mid E \text{ is a bounded to the left complex of Gorenstein injective } R\text{-modules and } E \simeq X \}.$$

2.3. Derived functors

The left derived tensor product functor $- \otimes_R \sim$ is computed by taking a projective resolution of the first argument or of the second one. Also, the right derived homomorphism functor $RHom_R(-, \sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one.

Let a be an ideal of R . The right derived functor of a -section functor $\Gamma_a(-) = \lim_{\rightarrow_n} Hom_R(\frac{R}{a^n}, -)$ is denoted by $R\Gamma_a(-)$. So for a given complex $X \in D_{\uparrow}(R)$, let I be an (every) injective resolution of X , then $R\Gamma_a(X) \simeq \Gamma_a(I)$. Also, for any integer i , the i -th local cohomology module of $X \in D_{\uparrow}(R)$ with respect to a is defined by $H_a^i(X) := H_{-i}(R\Gamma_a(X))$.

Following [16], for any two complexes $X \in D_{\uparrow}(R)$ and $Y \in D_{\uparrow}(R)$, we define $R\Gamma_a(X, Y) := R\Gamma_a(RHom_R(X, Y))$ and i -th generalized local cohomology module of $X \in D_{\uparrow}(R)$ and $Y \in D_{\uparrow}(R)$ with respect to a by $H_a^i(X, Y) := H_{-i}(R\Gamma_a(X, Y))$.

3. Results

We start this section by the following lemma, which is needed to prove the next theorem.

Lemma 3.1. *Let a be an ideal of R . Then every Gorenstein injective R -module is a Γ_a -coacyclic.*

Proof. See [14, Theorem 3.1]. Note that in the statement of that result, the ring is assumed to be Gorenstein. But, the author has not used this assumption. \square

The following theorem is a main result of our paper which gives a way to compute the right derived functor of a -section functor, which extends [14, Section 3] to local cohomology modules of complexes.

Theorem 3.2. *Let a be an ideal of R . Assume that $X \in D_{\square}(R)$ and E is a bounded to the left complex of Gorenstein injective R -modules such that $E \simeq X$. Then $R\Gamma_a(X) \simeq \Gamma_a(E)$, and so $H_a^i(X) = H_{-i}(\Gamma_a(E))$ for all $i \in \mathbb{Z}$. In particular, $-\inf R\Gamma_a(X) \leq \text{Gid}_R(X)$.*

Proof. Let I be an injective resolution of X . Then $E \simeq I$ and hence [1, 1.1.I and 1.4.I] yields the existence of a quasi-isomorphism $\alpha : E \rightarrow I$. Now, $\text{Cone}(\alpha)$ is an exact bounded to the left complex of Gorenstein injective modules. As any Gorenstein injective module is Γ_a -coacyclic by Lemma 3.1, by splitting $\text{Cone}(\alpha)$ into short exact sequences and using [2, Corollary 6.1.8 a)], one can see that $\Gamma_a(\text{Cone}(\alpha))$ is an exact sequence. Now, as [13, Lemma 2.4] asserts that $\Gamma_a(\text{Cone}(\alpha)) \simeq \text{Cone}(\Gamma_a(\alpha))$, we can deduce that $\text{Cone}(\Gamma_a(\alpha))$ is also exact. Therefore $\Gamma_a(\alpha) : \Gamma_a(E) \rightarrow \Gamma_a(I)$ is a quasi isomorphism by [2, Lemma A.1.19], and so $R\Gamma_a(X) \simeq \Gamma_a(I) \simeq \Gamma_a(E)$. So, $H_a^i(X) = H_{-i}(\Gamma_a(E))$ for all $i \in \mathbb{Z}$ and $-\inf R\Gamma_a(X) \leq \text{Gid}_R(X)$. \square

We need to the following definition and lemma for providing a corollary to Theorem 3.2.

Definition 3.3. A dualizing complex for R is a complex $D \in D_{\square}^f(R)$ such that the homothety morphism $R \rightarrow R\text{Hom}_R(D, D)$ is an isomorphism in $D(R)$ and D has finite injective dimension.

Lemma 3.4. *Assume that R possesses a dualizing complex and a is an ideal of R . Let M be an R -module. If M is Gorenstein injective, then $\Gamma_a(M)$ is also Gorenstein injective.*

Proof. By [11, Chapter V, Corollary 7.2], R has finite dimension and so the assertion follows from [15, Theorem 3.2]. \square

The following is an immediate corollary of Theorem 3.2 and Lemma 3.4 while was proven by a different method in [15, Corollary 3.3].

Corollary 3.5. *Assume that R possesses a dualizing complex and a is an ideal of R . For any $X \in D_{\downarrow}(R)$, we have $Gid_R R\Gamma_a(X) \leq Gid_R X$.*

Proof. Without loss of generality, we may and do assume that $Gid_R X$ is finite, say t . Then, there is a bounded Gorenstein injective resolution E with length t such that $E \simeq X$. So by Theorem 3.2, $R\Gamma_a(X) \simeq \Gamma_a(E)$ and by Lemma 3.4, $\Gamma_a(E)$ is a Gorenstein injective resolution of $R\Gamma_a(X)$ with maximal length t . Hence $Gid_R R\Gamma_a(X) \leq Gid_R X$. \square

The following proposition is needed in the proof of Theorem 3.7.

Proposition 3.6.

(I) *The class of Gorenstein flat R -modules is closed under direct sums.*

(II) *The class of Gorenstein injective R -modules is closed under direct products.*

(III) *Let M be a Gorenstein flat R -module and N be an injective R -module, then $\text{Hom}_R(M, N)$ is a Gorenstein injective R -module.*

(IV) *Let M be a Gorenstein flat R -module and N be a flat R -module. Then $M \otimes_R N$ is a Gorenstein flat R -module.*

(V) *Let M be a projective R -module and N be a Gorenstein injective R -module, then $\text{Hom}_R(M, N)$ is a Gorenstein injective R -module.*

Assume that R has a dualizing complex.

(VI) *Let M be a Gorenstein injective R -module and N be an injective R -module. Then $\text{Hom}_R(M, N)$ is a Gorenstein flat R -module.*

(VII) *Let M be a Gorenstein injective R -module and N be a flat R -module. Then $M \otimes_R N$ is a Gorenstein injective R -module.*

Proof. For the proofs of (I) and (II), see [12, Proposition 3.2 and Theorem 2.6], see [2, Theorem 6.4.2] for the proof of (III) and set $S := R$ in [3, Ascent table I (a), (e), (i) and (c)] for the proofs of (VI) and (VII). \square

Now, we present the following theorem, which gives some ways to compute generalized local cohomology modules.

Theorem 3.7. *Let a be an ideal of R . Assume that $X, Y \in D(R)$ are two non-homologically trivial complexes. The following assertions hold:*

(I) *Assume that $X \in D_{\downarrow}(R)$ and $Y \in D_{\uparrow}(R)$. Let P be a projective resolution of X and I be a bounded to the left complex of Gorenstein injective modules such that $I \simeq Y$. Then $R\Gamma_a(R\text{Hom}_R(X, Y)) \simeq \Gamma_a(\text{Hom}_R(P, I))$.*

(II) *Assume that $X \in D_{\downarrow}(R)$ and Q is a bounded to the right complex of Gorenstein flat R -modules such that $Q \simeq X$. If $Y \in D_{\uparrow}(R)$ and I is an injective resolution of Y , then $R\Gamma_a(R\text{Hom}(X, Y)) \simeq \Gamma_a(\text{Hom}(Q, I))$.*

Proof. (I) By Proposition 3.6 $\text{Hom}_R(G, N)$ is Gorenstein injective for all projective R -modules G and all Gorenstein injective R -modules N . Also, any direct product of Gorenstein injective R -modules is Gorenstein injective by Proposition 3.6 (II). So, $\text{Hom}_R(P, I)$ is a complex of Gorenstein injective R -modules. Now, as $\text{Hom}_R(P, I)$ is a bounded to the left complex by [2, Lemma A.2.2] and $R\text{Hom}_R(X, Y) \simeq \text{Hom}_R(P, I)$, Theorem 3.2 yields that $R\Gamma_a(R\text{Hom}_R(X, Y)) \simeq \Gamma_a(\text{Hom}_R(P, I))$.

(II) By using part (III) of Proposition 3.6 instead of part (V), the proof proceeds exactly as in (I), and so we leave it to the reader. \square

Lemma 3.8. *Let a be an ideal of R , $X \in D_{\downarrow}^f(R)$ and $Y \in D_{\square}(R)$. Then $R\Gamma_a(R\text{Hom}_R(X, Y)) \simeq R\text{Hom}_R(X, R\Gamma_a(Y))$.*

Proof. See [8, Proposition 6.1]. \square

We finish this paper by the following proposition that gives upper and lower bounds for vanishing of generalized local cohomology modules. We recall that for any ideal a of R and any complex X , $\text{depth}_R(a, X) := -\sup R\text{Hom}_R(\frac{R}{a}, X)$ and $\text{width}_R(a, X) := \inf(\frac{R}{a} \otimes_R^L X)$.

Proposition 3.9. *Assume that a is an ideal of R .*

(I) *Let $Y \in D_{\lceil}(R)$ and $X \in D_{\rceil}(R)$, then*

$$\sup R\Gamma_a(R\text{Hom}_R(X, Y)) \leq \sup Y - \text{width}_R(a, X).$$

Also, if R has a dualizing complex, then the following holds:

(II) *Let $X \in D_{\square}^f(R)$ be a non-homologically trivial complex such that either its projective or injective dimension is finite and $Y \in D_{\square}(R)$, then*

$$-\inf R\Gamma_a(R\text{Hom}_R(X, Y)) \leq \sup X + \text{Gid}_R Y.$$

Proof. (I) [2, Proposition A.4.6] asserts that $R\text{Hom}_R(X, Y) \in D_{\lceil}(R)$.

We have

$$\begin{aligned} \sup R\Gamma_a(R\text{Hom}_R(X, Y)) &= -\text{depth}_R(a, R\text{Hom}_R(X, Y)) \\ &= \sup R\text{Hom}_R\left(\frac{R}{a}, R\text{Hom}_R(X, Y)\right) \\ &= \sup R\text{Hom}_R\left(\frac{R}{a} \otimes_R^L X, Y\right) \\ &\leq \sup Y - \inf\left(\frac{R}{a} \otimes_R^L X\right) \\ &= \sup Y - \text{width}_R(a, X), \end{aligned}$$

where the first equality follows from [9, 2.4.1], the third equality follows from [2, A.4.21] and the inequality follows from [2, A.4.6.1].

Now, assume that R has a dualizing complex.

(II) Without loss of generality, we may and do assume that $\text{Gid}_R Y$ is finite. By Corollary 3.5, the Gorenstein injective dimension of $R\Gamma_a(Y)$ is finite, hence Lemma 3.8 and [4, Theorem 3.3] yield that

$$\begin{aligned}
-\inf R\Gamma_\alpha(R\text{Hom}_R(X, Y)) &= -\inf R\text{Hom}_R(X, R\Gamma_\alpha(Y)) \\
&\leq \sup X + \text{Gid}_R R\Gamma_\alpha(Y) \\
&\leq \sup X + \text{Gid}_R Y.
\end{aligned}$$

□

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